Image reconstruction from spectral information

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Basic Idea

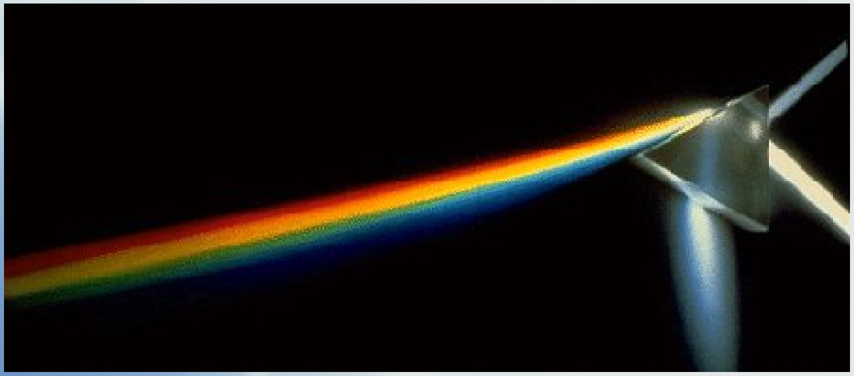
1. Create a regular grid on a bounded rectangular region of the plane. Each square of the grid represents a step function
2. We can compute analytically the Fourier transform of the function defined by the sum of these step functions. The coefficients are unaffected by this transformation
3. Compute the coefficients of the step functions using this fact and the sampled values of the spectral domain by a least-squares approximation

Signals and Frequency

- A signal can be decomposed in components called harmonics
- Harmonics reveal the “frequency contents” of the signal

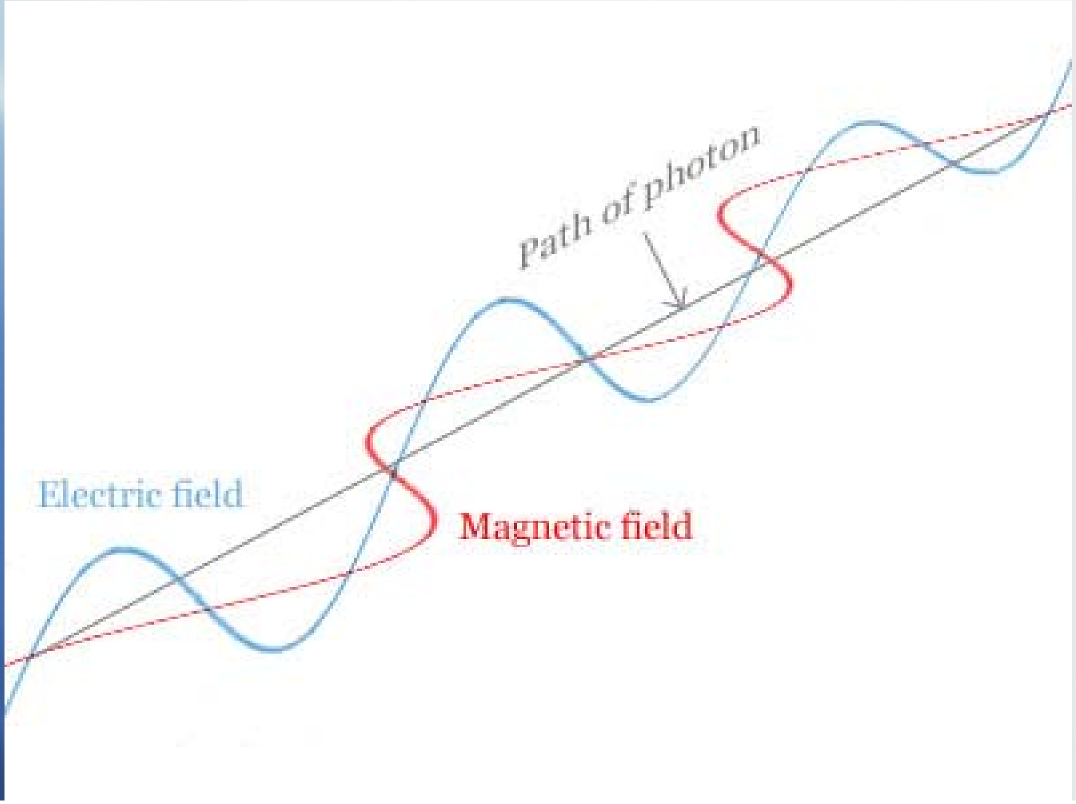
Signals and Frequency

- For example, light can be decomposed in its components with a prism



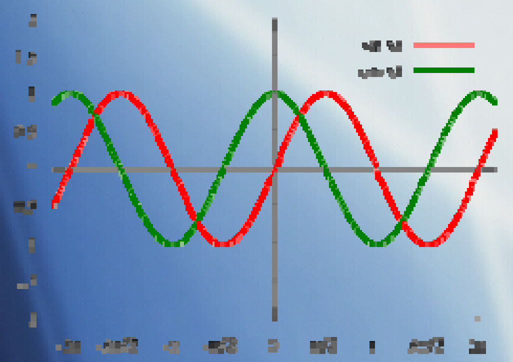
Signals and Frequency

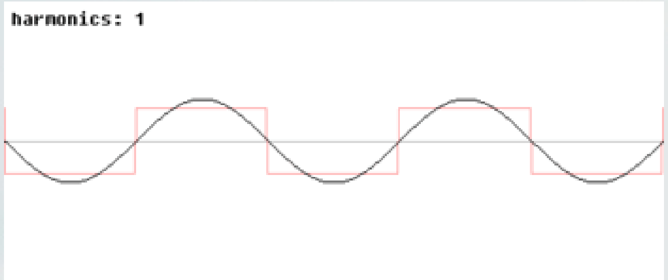
- The electromagnetic theory describes light as an electric and a magnetic wave



Signals and Frequency

- The basic harmonic functions can be built from the sine and cosine functions





The Fourier Transform

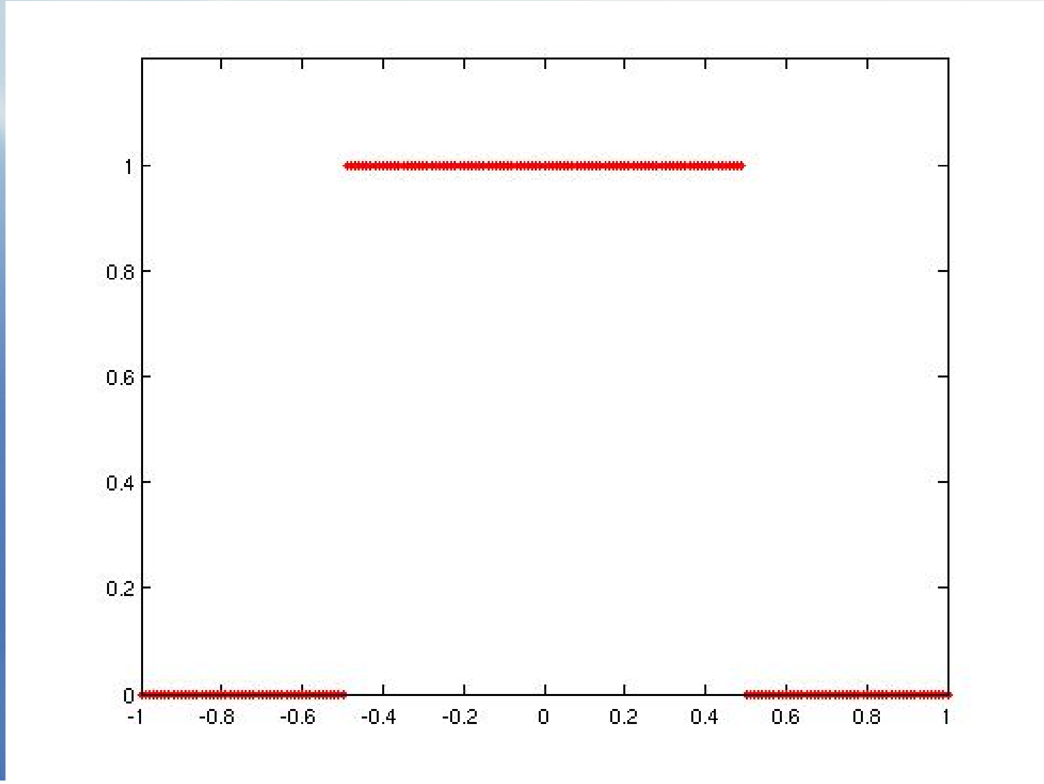
- We can use the Fourier transform to study the spectral contents of a signal

$$\hat{f}(\gamma) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x \gamma} dx$$

$$e^{-2\pi i \theta} = \cos(2\pi\theta) - i \sin(2\pi\theta)$$

The Fourier Transform

- Lets compute an example, the Fourier transform of the rectangle function



The Fourier Transform

- Lets compute an example, the Fourier transform of the rectangle function

$$\Pi(x) = \begin{cases} 1 & \text{if } |x| < \frac{1}{2}, \\ \frac{1}{2} & \text{if } |x| = \frac{1}{2}, \\ 0 & \text{if } |x| > \frac{1}{2}. \end{cases}$$

The Fourier Transform

$$
\begin{aligned}
\widehat{\Pi}(\gamma) &= \int_{-\infty}^{\infty} \Pi(x) e^{-2\pi i x \gamma} dx \\
&= \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-2\pi i x \gamma} dx \\
&= \frac{1}{-2\pi i \gamma} e^{-2\pi i x \gamma} \Big|_{x=-\frac{1}{2}}^{\frac{1}{2}} \\
&= \frac{1}{-2\pi i \gamma} \left(e^{-\pi i \gamma} - e^{\pi i \gamma} \right) \\
&= \frac{\sin(\pi\gamma)}{\pi\gamma} := \operatorname{sinc}_{\pi}(\gamma).
\end{aligned}
$$

The Fourier Transform

- We can also extend the definition of the Fourier transform to the plane

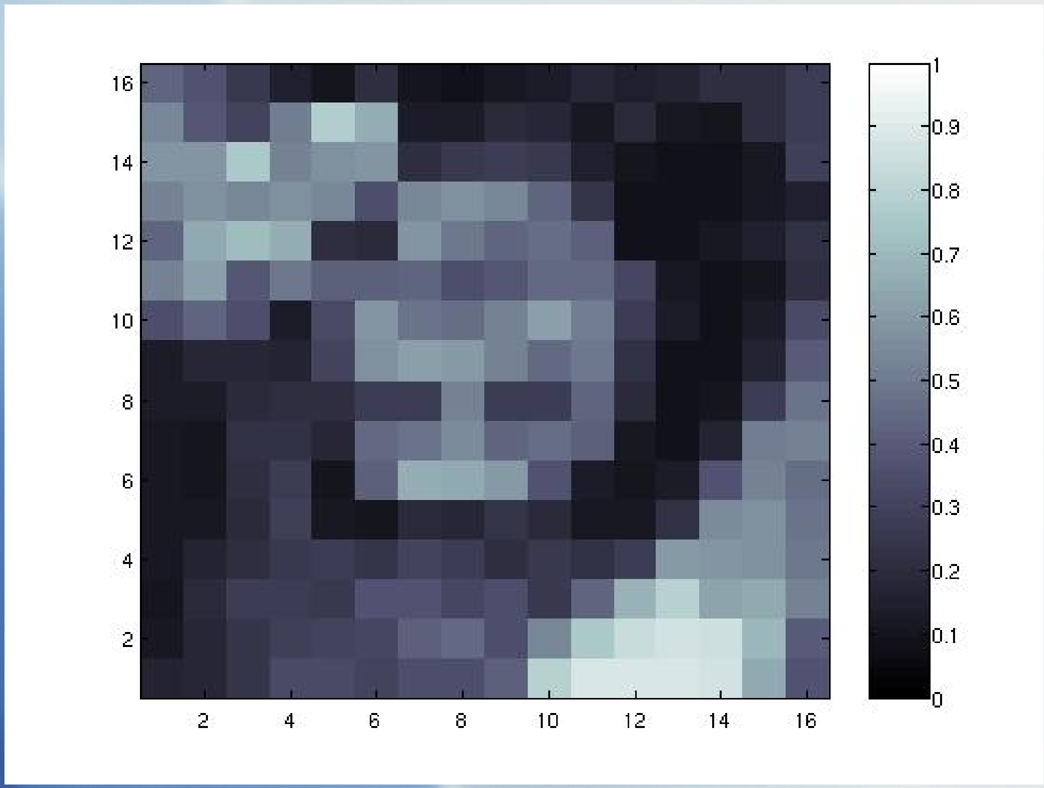
$$\hat{f}(\sigma,\gamma)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)e^{-2\pi i(x\sigma+y\gamma)}dxdy$$

What is an image?

- Suppose you have a black and white image made of “pixels”
- Suppose the image has N by N pixels
- Assume each pixel has a gray value

What is an image?

- For example: 16 by 16 with 256 shades



What is an image?

- Mathematically, we can think of an image as a sum of characteristic functions in the plane

What is an image?

$$f(x,y) = \sum_{k,l} f_{k,l}\chi_{k,l}(x,y)$$

$$\chi_{k,l} = \begin{cases} 1 & \text{if } (x,y) \in \square_{k,l}, \\ 0 & \text{otherwise.} \end{cases}$$

Fourier Transform of an image

$$\begin{aligned}\hat{f}(\sigma,\gamma) &= \int_{\mathbb{R}^2} \sum_{k,l} f_{k,l}\chi_{k,l}(x,y) e^{-2\pi i(x\sigma+y\gamma)} dxdy \\ &= \sum_{k,l} \int_{\mathbb{R}^2} f_{k,l}\chi_{k,l}(x,y) e^{-2\pi i(x\sigma+y\gamma)} dxdy \\ &= \sum_{k,l} \int_l^{l+1} \int_k^{k+1} f_{k,l} e^{-2\pi i(x\sigma+y\gamma)} dxdy \\ &= \sum_{k,l} f_{k,l} \int_k^{k+1} e^{-2\pi i x\sigma} dx \int_l^{l+1} e^{-2\pi i y\gamma} dy,\end{aligned}$$

$$\int_k^{k+1} e^{-2\pi i x\sigma} dx = \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-2\pi i(x'+k+\frac{1}{2})\sigma} dx'$$

Fourier Transform of an image

- From which we obtain,

$$
\begin{aligned}
\hat{f}(\sigma,\gamma) &= \sum_{k,l} f_{k,l} \int_k^{k+1} e^{-2\pi i x\sigma} dx \int_l^{l+1} e^{-2\pi i y\gamma} dy \\
&= \sum_{k,l} f_{k,l} \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-2\pi i (x'+k+\frac{1}{2})\sigma} dx' \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-2\pi i (y'+l+\frac{1}{2})\gamma} dy' \\
&= \sum_{k,l} f_{k,l} \left(e^{-2\pi i (k+\frac{1}{2})\sigma} \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-2\pi i x'\sigma} dx' \right) \left(e^{-2\pi i (l+\frac{1}{2})\gamma} \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-2\pi i y'\gamma} dy' \right) \\
&= \sum_{k,l} f_{k,l} e^{-2\pi i (k+\frac{1}{2})\sigma} \mathrm{sinc}_\pi(\sigma) e^{-2\pi i (l+\frac{1}{2})\gamma} \mathrm{sinc}_\pi(\gamma).
\end{aligned}
$$

Fourier Transform of an image

- Or, in simplified form,

$$\hat{f}(\sigma,\gamma) = \left(\sum_{k,l} f_{k,l} e^{-2\pi i k\sigma} e^{-2\pi i l\gamma}\right) e^{-i\pi(\sigma+\gamma)} \mathrm{sinc}_\pi(\sigma)\mathrm{sinc}_\pi(\gamma)$$

$$\frac{\hat{f}(\sigma+m,\gamma+n)}{\hat{f}(\sigma,\gamma)} = (-1)^{m+n} \frac{\mathrm{sinc}_\pi(\sigma+m)\mathrm{sinc}_\pi(\gamma+n)}{\mathrm{sinc}_\pi(\sigma)\mathrm{sinc}_\pi(\gamma)}$$

$$\hat{f}(\sigma+m,\gamma+n) = (-1)^{m+n} \frac{\mathrm{sinc}_\pi(\sigma+m)\mathrm{sinc}_\pi(\gamma+n)}{\mathrm{sinc}_\pi(\sigma)\mathrm{sinc}_\pi(\gamma)} \hat{f}(\sigma,\gamma)$$

Fourier Transform of an image

- This means we only need to sample on the unit square [0,1)x[0,1) in the spectral domain!
- Therefore, we can collect all samples in the unit square and associate them to a regular grid that is sufficiently refined

Inversion theory

- Having developed all this theory, how do we actually recover an image from samples in the frequency domain?

Inversion theory

- Lets re-enumerate the squares in the image by an alphabetical order

$$f(x,y) = \sum_j f_j \chi_j(x,y)$$

Inversion theory

- The Fourier transform of the image becomes

$$\hat{f}(\sigma,\gamma) = \left(\sum_j f_j e^{-2\pi i k_j \sigma} e^{-2\pi i l_j \gamma}\right) e^{-i\pi(\sigma+\gamma)} \mathrm{sinc}_\pi(\sigma) \mathrm{sinc}_\pi(\gamma)$$

Inversion theory

- So, assume we have samples of the Fourier transform of the image, then we should satisfy

$$\hat{f}(\sigma_i, \gamma_i) = \sum_{j=0}^{N^2-1} f_j a_j(\sigma_i, \gamma_i) = b_i$$

$$a_j(\sigma, \gamma) = e^{-2\pi i k_j \sigma} e^{-2\pi i l_j \gamma} e^{-i\pi(\sigma+\gamma)} \mathrm{sinc}_\pi(\sigma) \mathrm{sinc}_\pi(\gamma)$$

Inversion theory

- We identify a matrix vector product!

$$\begin{pmatrix} a_0(\sigma_0,\gamma_0) & a_1(\sigma_0,\gamma_0) & \dots & a_{N^2-1}(\sigma_0,\gamma_0) \\ a_0(\sigma_1,\gamma_1) & a_1(\sigma_1,\gamma_1) & \dots & a_{N^2-1}(\sigma_1,\gamma_1) \\ \vdots & \vdots & & \vdots \\ a_0(\sigma_{M-1},\gamma_{M-1}) & a_1(\sigma_{M-1},\gamma_{M-1}) & \dots & a_{N^2-1}(\sigma_{M-1},\gamma_{M-1}) \end{pmatrix} \begin{pmatrix} f_0 \\ f_1 \\ \vdots \\ f_{N^2-1} \end{pmatrix} = \begin{pmatrix} b_0 \\ b_1 \\ \vdots \\ b_{M-1} \end{pmatrix}$$

$$\boldsymbol{Af} = \boldsymbol{b}$$

Inversion theory

- A is M by N^2, f is N^2 by 1, and b is M by 1
- M needs to be bigger than N^2
- A can become really big if N is big
- Storage problem

Inversion theory

- Lets try the following trick, and explore some consequences of this new approach

$$\boldsymbol{A}\boldsymbol{f} = \boldsymbol{b}$$

$$(\boldsymbol{A}^*\boldsymbol{A})\boldsymbol{f} = \boldsymbol{A}^*\boldsymbol{b}$$

$$\boldsymbol{f} = (\boldsymbol{A}^*\boldsymbol{A})^{-1}\boldsymbol{A}^*\boldsymbol{b}$$

Inversion theory

- Solving the modified system is equivalent to solving the least squares problem and the solution f will satisfy

$$\|\boldsymbol{A}\boldsymbol{f} - \boldsymbol{b}\|^2 = \min_{\boldsymbol{x}} \|\boldsymbol{A}\boldsymbol{x} - \boldsymbol{b}\|^2$$

Algorithms for inversion

- We know then that we have a least squares problem
- We have at least two good methods to try:
- Singular Value Decomposition
- Conjugate Gradient method

Algorithms for inversion

- Singular Value Decomposition

$$A = U\Sigma V^*$$

$$\begin{aligned} A^*A &= (U\Sigma V^*)^* U\Sigma V^* \\ &= (V\Sigma U^*) U\Sigma V^* \\ &= V\Sigma^2 V^* \end{aligned}$$

Algorithms for inversion

- We form the residue

$$\boldsymbol{r} = \boldsymbol{A}\boldsymbol{x} - \boldsymbol{b}$$

$$\begin{aligned} \|\boldsymbol{r}\|^2 &= \boldsymbol{r}^*\boldsymbol{r} \\ &= (\boldsymbol{U}^*\boldsymbol{r})^*(\boldsymbol{U}^*\boldsymbol{r}) \\ &= \|\boldsymbol{U}^*\boldsymbol{r}\|^2 \\ &= \|\boldsymbol{U}^*\boldsymbol{A}\boldsymbol{x} - \boldsymbol{U}^*\boldsymbol{b}\|^2 \\ &= \|\boldsymbol{\Sigma}\boldsymbol{V}^*\boldsymbol{x} - \boldsymbol{c}\|^2 \end{aligned}$$

Algorithms for inversion

- The problem though is space
- Space to store matrix grows like N^4
- Try Conjugate Gradient method instead
- We know A*A is Hermitian and positive semidefinite, good candidate for CG!

Experiments

- We will create a bank of images to sample in the spectral domain
- Test the CG algorithm to reconstruct the original image
- Compare with downsampled original

References

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- John J. Benedetto and Paulo J. S. G. Ferreira. Moderm Sampling Theory: Mathematics and Applications. Birkhauser, 2001.

References

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Original
64x64 Reconstruction
32x32 Reconstruction
16x16 Reconstruction